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On weighted q-Čebyšev–Grüss type inequalities

Wengui Yang *

Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China

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ABSTRACT

In this paper, weighted q-Čebyšev–Grüss type inequalities are established by using the weighted q-integral Montgomery identity. Some applications are also obtained.

1. Introduction and preliminaries

The well-known Grüss integral inequality can be stated as follows (see [1,2]):

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x-\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right)\right|\le\frac{1}{4}(P-p)(Q-q),\tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ such that $p\le f(x)\le P$, $q\le g(x)\le Q$, for all $x\in[a,b]$, where p,P,q,Q are real constants.

By using Pěcarić's extension of the Montgomery identity, Pachpatte gave the weighted Čebyšev–Grüss type inequality as follows (see [3]): define $\|\ \|_\infty$ as the usual Lebesgue norm on $L_\infty[a,b]$, that is, $\|h\|_\infty:=\operatorname{ess\,sup}_{t\in[a,b]}|h(t)|$ for $h\in L_\infty[a,b]$.

Let $f,g:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f',g':[a,b]\to\mathbb{R}$ are integrable on $[a,b]$. Let $\omega:[a,b]\to[0,\infty)$ be an integrable function satisfying $\int_a^b\omega(t)\mathrm{d}t=1$. Then

$$|T(\omega,f,g)|\le\|f'\|_\infty\|g'\|_\infty\int_a^b\omega(x)H^2(x)\mathrm{d}x,\tag{1.2}$$

where

$$T(\omega,f,g)=\int_a^b\omega(x)f(x)g(x)\mathrm{d}x-\left(\int_a^b\omega(x)f(x)\mathrm{d}x\right)\left(\int_a^b\omega(x)g(x)\mathrm{d}x\right),$$

and

$$H(x)=\int_a^b|P_\omega(x,t)|\mathrm{d}t$$

* Tel.: +86 15939854723.
E-mail address: wgyang0617@yahoo.com.

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for all $x \in [a, b]$ and $P_\omega(x, t)$ is the weighted Peano kernel given by

$$P_\omega(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x \le t \le b, \end{cases}$$

where $W(t) = \int_a^t \omega(x)\mathrm{d}x$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$.

In the following some definitions and propositions are cited on q-integral in order to narrate conveniently. Some details see [4,5].

In what follows, q is a real number satisfying $0 < q < 1$.

Definition 1 (*See [5]*)**.** For an arbitrary function $f(x)$, the q-differential is defined by $(\mathrm{d}_q f)(x) := f(qx) - f(x)$. In particular, $\mathrm{d}_q x = (q-1)x$. The q-derivative is defined by

$$(D_q f)(x) := \frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x} = \frac{f(qx) - f(x)}{(q-1)x}.$$

Clearly, if $f(x)$ is differentiable, then $\lim_{q\to 1}(D_q f)(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$.

Definition 2 (*See [5]*)**.** Suppose $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(x)\mathrm{d}_q x := (1-q)b\sum_{j=0}^{\infty} q^j f(bq^j), \tag{1.3}$$

and

$$\int_a^b f(x)\mathrm{d}_q x := \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x. \tag{1.4}$$

The definite q-integral defined above is too general for our purpose of studying inequalities. For example, if $f(x) \ge 0$, it is not necessarily true that $\int_a^b f(x)\mathrm{d}_q x \ge 0$.

From now on, we will use a special type of the definite q-integral, which we will call the restricted definite q-integral. Throughout all the paper, we will use the following notations:

$$c_j = bq^j, \quad \text{for } j \in \{0, 1, \ldots, n\}, \qquad a = c_n = bq^n.$$

Definition 3 (*See [5]*)**.** Let $0 < q < 1$, $b > 0$, and $n \in \mathbb{Z}^+$. The restricted q-integral is defined as $\int_{bq^n}^b f(x)\mathrm{d}_q x$.

The following formula readily follows from (1.3) and (1.4):

$$\int_a^b f(x)\mathrm{d}_q x = \int_{bq^n}^b f(x)\mathrm{d}_q x = (1-q)b\sum_{j=0}^{n-1} q^j f(bq^j) = (1-q)\sum_{j=0}^{n-1} c_j f(c_j). \tag{1.5}$$

Note that the restricted integral $\int_a^b f(x)\mathrm{d}_q x$ is just a finite sum, so no questions about convergency arise. It is easy to check that

$$\int_a^b D_q f\,\mathrm{d}_q x = f(b) - f(a). \tag{1.6}$$

Obviously, if $f(x) \ge g(x)$ on $[a, b]$, then $\int_a^b f(x)\mathrm{d}_q x \ge \int_a^b g(x)\mathrm{d}_q x$. If $0 < k < n$, then

$$\int_a^b f(x)\mathrm{d}_q x = \int_a^{c_k} f(x)\mathrm{d}_q x + \int_{c_k}^b f(x)\mathrm{d}_q x. \tag{1.7}$$

The following is the formula for the q-integration by parts:

$$\int_a^b f(x)(D_q g)(x)\mathrm{d}_q x = [f(x)g(x)]_a^b - \int_a^b g(qx)(D_q f)(x)\mathrm{d}_q x. \tag{1.8}$$

Cauciman [5] gave the q-integral Grüss's inequality as follows:

Assume that $p \le f(x) \le P$, $q \le g(x) \le Q$, for all $x \in [a, b]$, where p, P, q, Q are real constants. Then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}_q x - \left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q x\right)\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}_q x\right)\right| \le \frac{1}{4}(P-p)(Q-q).$$

In this paper, by using the weighted q-integral Montgomery identity, weighted q-Čebyšev–Grüss type inequalities are established similarly to inequality (1.2). Some applications are also obtained.

2. Main results

Assume that $\omega : [a, b] \to [0, \infty)$ satisfying $\int_a^b \omega(x) \mathrm{d}_q x = 1$. Set $W(t) = \int_a^t \omega(x) \mathrm{d}_q x$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. We give the weighted q-integral Peano kernel $P_\omega(x, t)$ defined by

$$P_\omega(x, t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x \le t \le b. \end{cases} \tag{2.1}$$

We use the following notation to simplify details of the presentation. For some suitable functions $\omega, f, g : [a, b] \to \mathbb{R}$, we set

$$T(\omega, f, g) := \int_a^b \omega(x) f(qx) g(qx) \mathrm{d}_q x - \left(\int_a^b \omega(x) f(qx) \mathrm{d}_q x \right) \left(\int_a^b \omega(x) g(qx) \mathrm{d}_q x \right),$$

and define $\|\cdot\|$ as $\|h\| := \sup_{t \in [a,b]} |h(t)|$ for $h \in [a, b]$.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$. And $\omega : [a, b] \to [0, \infty)$ satisfying $\int_a^b \omega(x) \mathrm{d}_q x = 1$, then*

$$f(x) = \int_a^b \omega(t) f(qt) \mathrm{d}_q t + \int_a^b P_\omega(x, t)(D_q f)(t) \mathrm{d}_q t, \tag{2.2}$$

for $x \in [a, b]$, where the weighted q-integral Peano kernel $P_\omega(x, t)$ is defined by (2.1).

Proof. According to the weighted q-integral Peano kernel $P_\omega(x, t)$, we obtain

$$\begin{aligned} \int_a^b P_\omega(x, t)(D_q f)(t) \mathrm{d}_q t &= \int_a^x W(t)(D_q f)(t) \mathrm{d}_q t + \int_x^b (W(t) - 1)(D_q f)(t) \mathrm{d}_q t \\ &= \int_a^b W(t)(D_q f)(t) \mathrm{d}_q t - \int_x^b (D_q f)(t) \mathrm{d}_q t \\ &= [W(t) f(t)]_a^b - \int_a^b f(qt)(D_q W)(t) \mathrm{d}_q t - [f(b) - f(x)]. \end{aligned}$$

Due to the definitions of q-derivative and q-integral, one has

$$(D_q W)(t) = \frac{\int_a^{qt} \omega(x) \mathrm{d}_q x - \int_a^t \omega(x) \mathrm{d}_q x}{(q - 1)t}. \tag{2.3}$$

By using the formula (1.5), we have

$$\int_a^t \omega(x) \mathrm{d}_q x - \int_a^{qt} \omega(x) \mathrm{d}_q x = \int_{qt}^t \omega(x) \mathrm{d}_q x = (1 - q) t \omega(t). \tag{2.4}$$

Combining (2.3), (2.4) and the above equality, we obtain

$$\begin{aligned} \int_a^b P_\omega(x, t)(D_q f)(t) \mathrm{d}_q t &= [W(t) f(t)]_a^b - \int_a^b f(t)(D_q W)(t) \mathrm{d}_q t - [f(b) - f(x)] \\ &= f(b) - \int_a^b \omega(t) f(qt) \mathrm{d}_q t - [f(b) - f(x)]. \end{aligned}$$

Thus

$$f(x) = \int_a^b \omega(t) f(qt) \mathrm{d}_q t + \int_a^b P_\omega(x, t)(D_q f)(t) \mathrm{d}_q t,$$

and this completes the proof. □

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$. And $\omega : [a, b] \to [0, \infty)$ satisfying $\int_a^b \omega(x) \mathrm{d}_q x = 1$, then*

$$|T(\omega, f, g)| \le \|D_q f\| \, \|D_q g\| \int_a^b \omega(x) H^2(qx) \mathrm{d}_q x, \tag{2.5}$$

where

$$H(x) = \int_a^b |P_\omega(x, t)| \mathrm{d}_q t$$

for all $x \in [a, b]$, the weighted q-integral Peano kernel $P_\omega(x, t)$ is defined by (2.1).

Proof. Since the functions f and g satisfy the hypothesis of Theorem 1, the following identities hold:

$$f(x) = \int_a^b \omega(t)f(qt)\mathrm{d}_qt + \int_a^b P_\omega(x,t)(D_qf)(t)\mathrm{d}_qt, \tag{2.6}$$

and

$$g(x) = \int_a^b \omega(t)g(qt)\mathrm{d}_qt + \int_a^b P_\omega(x,t)(D_qg)(t)\mathrm{d}_qt. \tag{2.7}$$

Due to the above two inequalities, we have

$$f(qx) = \int_a^b \omega(t)f(qt)\mathrm{d}_qt + \int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt, \tag{2.8}$$

and

$$g(qx) = \int_a^b \omega(t)g(qt)\mathrm{d}_qt + \int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt. \tag{2.9}$$

Using (2.8) and (2.9), we obtain

$$\begin{aligned}&\left(f(qx) - \int_a^b \omega(t)f(qt)\mathrm{d}_qt\right)\left(g(qx) - \int_a^b \omega(t)g(qt)\mathrm{d}_qt\right)\\ &\quad= \left(\int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right)\left(\int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right).\end{aligned}$$

Consequently,

$$\begin{aligned}&f(qx)g(qx) - g(qx)\int_a^b \omega(t)f(qt)\mathrm{d}_qt - f(qx)\int_a^b \omega(t)g(qt)\mathrm{d}_qt + \int_a^b \omega(t)f(qt)\mathrm{d}_qt\int_a^b \omega(t)g(qt)\mathrm{d}_qt\\ &\quad= \left(\int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right)\left(\int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right).\end{aligned}$$

Multiplying both sides by $\omega(x)$ and then q-integrating the resultant identity with respect to x from a to b, we get

$$T(\omega,f,g) = \int_a^b \omega(x)\left(\int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right)\left(\int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right)\mathrm{d}_qx.$$

Finally, using the properties of modulus we observe that

$$|T(\omega,f,g)| \le \|D_qf\|\,\|D_qg\|\int_a^b \omega(x)H^2(qx)\mathrm{d}_qx.$$

The proof of Theorem 2 is complete. □

Theorem 3. *Let $f, g : [a,b] \to \mathbb{R}$. And $\omega : [a,b] \to [0,\infty)$ satisfying $\int_a^b \omega(x)\mathrm{d}_qx = 1$, then*

$$|T(\omega,f,g)| \le \frac{1}{2}\int_a^b \omega(x)[|g(qx)|\,\|D_qf\| + |f(qx)|\,\|D_qg\|]H(qx)\mathrm{d}_qx,$$

where $H(x)$ is defined in Theorem 2.

Proof. Multiplying both sides of (2.8) and (2.9) by $\omega(x)g(qx)$ and $\omega(x)f(qx)$, adding the resulting identities and rewriting we have

$$\begin{aligned}\omega(x)f(qx)g(qx) &= \frac{1}{2}\left(\omega(x)g(qx)\int_a^b \omega(t)f(qt)\mathrm{d}_qt + \omega(x)f(qx)\int_a^b \omega(qt)g(qt)\mathrm{d}_qt\right)\\ &\quad+\frac{1}{2}\left(\omega(x)g(qx)\int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt + \omega(x)f(qx)\int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right).\end{aligned}$$

Q-integrating both sides of the above with respect to x from a to b and rewriting we have

$$T(\omega,f,g) = \frac{1}{2}\int_a^b \left(\omega(x)g(qx)\int_a^b P_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt + \omega(x)f(qx)\int_a^b P_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right)\mathrm{d}_qx.$$

From the above and using the properties of modulus we observe that

$$\begin{aligned}|T(\omega,f,g)| &\le \frac{1}{2}\int_a^b\left(\omega(x)|g(qx)|\int_a^b|P_\omega(qx,t)|\,|(D_qf)(t)|\mathrm{d}_qt+\omega(x)|f(qx)|\int_a^b|P_\omega(qx,t)||(D_qg)(t)|\mathrm{d}_qt\right)\mathrm{d}_qx\\ &\le \frac{1}{2}\int_a^b\omega(x)[|g(qx)|\,\|D_qf\|+|f(qx)|\,\|D_qg\|]H(qx)\mathrm{d}_qx.\end{aligned}$$

The proof of Theorem 3 is complete. □

3. Some applications

For some given functions $f, g : [a, b] \to \mathbb{R}$,

$$S(\omega,f,g):=f(qx)g(qx)-\frac{1}{2}\left(f(qx)\int_a^b\omega(t)g(qt)\mathrm{d}_qt+g(qx)\int_a^b\omega(t)f(qt)\mathrm{d}_qt\right). \tag{3.1}$$

Theorem 4. *Let $f, g : [a, b] \to \mathbb{R}$. And $\omega : [a, b] \to [0, \infty)$ satisfying $\int_a^b \omega(x)\mathrm{d}_qx = 1$, then*

$$|S(\omega,f,g)|\le\frac{1}{2}[|f(qx)|\,\|D_qg\|+|g(qx)|\,\|D_qf\|]H(qx),$$

for $x \in [a, b]$ and

$$|T(\omega,f,g)|\le\frac{1}{2}[\|f(qx)\|\,\|D_qg\|+\|g(qx)\|\,\|D_qf\|]\int_a^b\omega(x)H(qx)\mathrm{d}_qx,$$

where $H(x)$ is defined in Theorem 2.

Proof. Multiplying both sides of (2.8) and (2.9) by $g(qx)$ and $f(qx)$, adding the resulting identities and rewriting we have

$$\begin{aligned}f(qx)g(qx) &= \frac{1}{2}\left(f(qx)\int_a^b\omega(t)g(qt)\mathrm{d}_qt+g(qx)\int_a^b\omega(t)f(qt)\mathrm{d}_qt\right)\\ &\quad+\frac{1}{2}\left(f(qx)\int_a^bP_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt+g(qx)\int_a^bP_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right),\end{aligned} \tag{3.2}$$

which implies

$$S(\omega,f,g)=\frac{1}{2}\left(f(qx)\int_a^bP_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt+g(qx)\int_a^bP_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right).$$

We observe

$$|S(\omega,f,g)|\le\frac{1}{2}[|f(qx)|\,\|D_qg\|+|g(qx)|\,\|D_qf\|]H(qx).$$

Multiplying both sides of (3.2) by $\omega(x)$ and q-integrating with respect to x from a to b, we have

$$T(\omega,f,g)=\frac{1}{2}\left(\int_a^b\left(\omega(x)f(qx)\int_a^bP_\omega(qx,t)(D_qg)(t)\mathrm{d}_qt\right)\mathrm{d}_qx+\int_a^b\left(\omega(x)g(qx)\int_a^bP_\omega(qx,t)(D_qf)(t)\mathrm{d}_qt\right)\mathrm{d}_qx\right),$$

which implies

$$|T(\omega,f,g)|\le\frac{1}{2}[\|f(qx)\|\,\|D_qg\|+\|g(qx)\|\,\|D_qf\|]\int_a^b\omega(x)H(qx)\mathrm{d}_qx.$$

The proof of Theorem 4 is complete. □

Theorem 5. *Let $f, g : [a, b] \to \mathbb{R}$. And $\omega : [a, b] \to [0, \infty)$ satisfying $\int_a^b \omega(x)\mathrm{d}_qx = 1$, then*

$$|T(\omega,f,g)|\le\|g(qx)\|\,\|D_qf\|\int_a^b\omega(x)H(qx)\mathrm{d}_qx, \tag{3.3}$$

and

$$|T(\omega,f,g)|\le\|f(qx)\|\,\|D_qg\|\int_a^b\omega(x)H(qx)\mathrm{d}_qx, \tag{3.4}$$

where $H(x)$ is defined in Theorem 2.

Proof. We prove only (3.3), since the proof of (3.4) is similar. The identity (2.8), shows that

$$f(qx) = \int_a^b \omega(t) f(qt) \mathrm{d}_q t + \int_a^b P_\omega(qx, t)(D_q f)(t) \mathrm{d}_q t, \tag{3.5}$$

for $x \in [a, b]$.

Now, if we multiply (3.5) by $\omega(x)g(qx)$ and q-integrate over $x \in [a, b]$, we deduce

$$\int_a^b \omega(x) f(qx) g(qx) \mathrm{d}_q x = \int_a^b \omega(t) f(qt) \mathrm{d}_q t \int_a^b \omega(x) g(qx) \mathrm{d}_q x + \int_a^b \omega(x) g(qx) \left(\int_a^b P_\omega(qx, t)(D_q f)(t) \mathrm{d}_q t \right) \mathrm{d}_q x,$$

which provides another representation for the functional $T(\omega, f, g)$ namely,

$$T(\omega, f, g) = \int_a^b \omega(x) g(qx) \left(\int_a^b P_\omega(qx, t)(D_q f)(t) \mathrm{d}_q t \right) \mathrm{d}_q x. \tag{3.6}$$

From (3.6) and using the properties of modulus we observe that (3.3). □

Remark. If $q \to 1^-$, by Definitions 1 and 2, the q-derivative and q-integral are the usual derivative and integral, so our results are reduced to some results in [3,6].

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